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On an L_p approach to the stationary and nonstationary problems of the Ginzburg–Landau–Maxwell equations

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Abstract

In this paper, we study a stationary and a nonstationary problem of the Ginzburg–Landau–Maxwell equations with Coulomb gauge in the L_p framework. First we prove a unique existence of stationary solution near the constant state with a small external magnetic field. Moreover, we prove a globally in time existence of solutions to the time dependent Ginzburg–Landau–Maxwell equations with small initial data and external magnetic field, and we show its convergence to the corresponding stationary solution when time tends to infinity. The key of our approach is to use various L_p–L_q estimates of the analytic semigroup generated by the linearized problem. Especially our initial data belong to L_3 without any additional regularity.

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1. Introduction

In this paper we consider the time dependent Ginzburg–Landau–Maxwell equations with Coulomb gauge:

$$(\rho_t - i\Phi\rho) = (\nabla - i\mathbf{A})^2\rho + \kappa\big(1-|\rho|^2\big)\rho \qquad \text{in } \Omega\times\mathbb{R}_+, \tag{1.1}$$

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$$\eta(\mathbf{A}_t - \nabla\Phi) + \nabla \times (\nabla \times \mathbf{A}) + \nabla \times \mathbf{H} = \mathbf{J}_{GL}(\rho, \mathbf{A}) \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.2}$$

$$\mathbf{J}_{\mathrm{GL}}(\rho, \mathbf{A}) = -\frac{i}{2}\Big[\bar{\rho}(\nabla - i\mathbf{A})\rho - \rho\overline{(\nabla - i\mathbf{A})\rho}\Big], \tag{1.3}$$

$$\nabla \cdot \mathbf{A} = 0 \quad \text{(Coulomb gauge)} \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.4}$$

$$\rho(x, 0) = \rho_0(x), \qquad \mathbf{A}(x, 0) = \mathbf{A}_0(x), \qquad \text{in } \Omega, \tag{1.5}$$

$$\partial_\nu \rho|_\Gamma = 0, \qquad \nu \cdot \mathbf{A}|_\Gamma = 0, \qquad (\nabla \times \mathbf{A} + \mathbf{H}) \times \nu|_\Gamma = 0, \tag{1.6}$$

where $\mathbb{R}_+ = (0, \infty)$; $\rho(x)$, $\Phi(x)$ and $\mathbf{A}(x) = {}^T(A_1(x), A_2(x), A_3(x))$[1] are the complex-valued order parameter, the scalar electric potential and the magnetic vector potential, respectively; η and κ are positive constants called the *conductivity* and the *Ginzburg–Landau parameter of substance*, respectively; $\mathbf{H}(x) = {}^T(H_1(x), H_2(x), H_3(x))$ is the external magnetic field; $\bar{\rho}$ is the complex conjugate of ρ; $\mathbf{J}_{\mathrm{GL}}(\rho, \mathbf{A})$ is called the *Ginzburg–Landau current*; Ω is a domain in the three dimensional Euclidean space $\mathbb{R}^3$ which satisfies one of the following assumptions:

(i) Ω is bounded.
(ii) Ω is an exterior domain, i.e., a domain having a compact nonempty complement.
(iii) Ω is a perturbed half space, i.e., there exists some open ball B such that $\Omega \setminus B = \mathbb{R}^3_+ \setminus B$, where

$$\mathbb{R}^3_+ = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 > 0\};$$

$\Gamma = \partial\Omega$ denotes the boundary of Ω which is a $C^{2,1}$ hypersurface; and $\nu = {}^T(\nu_1, \nu_2, \nu_3)$ is the unit outer normal vector to the boundary Γ. In this paper, we call the system (1.1)–(1.6) the TDGLM equations, which were proposed by Schmid [9] or Gor'kov and Eliashberg [7]. It describes a gauge invariant variant of the gradient flow of the minimizing problem of the Gibbs free energy for the order parameter ρ. They also proposed the time dependent Ginzburg–Landau equations coupled with the hyperbolic Maxwell equations.

The TDGLM was treated first in the L_2 framework as far as the author knows. In fact, Klimov [5] investigated the existence of multiple weak solution with the assumption that external magnetic field is not applied. Tsutsumi, Kasai and Oishi [11] established the existence of nontrivial stationary solution in the presence of an external magnetic field for any $\kappa > 0$. But, to show the stability of such nontrivial stationary solution in the L_2 framework, some regularity assumption on the initial data was necessary. In order to avoid such regularity assumption on the initial data, according to the well-known argument due to Kato concerning the Navier–Stokes equation, it is better to treat the TDGLM in the L_p framework. From this point of view, Akiyama et al. [1] studied (1.1)–(1.6) in the L_p framework without any regularity assumptions on the initial data. They studied the case where the external magnetic field $\mathbf{H}$ depends on (t, x) and they proved a globally in time existence of solutions to TDGLM provided that some norms of $\mathbf{H}(t, x)$, $\mathbf{H}_t(t, x)$ and initial data are small enough. Moreover, roughly speaking, they proved that the asymptotic behavior of solutions is the same as that of $\mathbf{H}(t, x)$ and $\mathbf{H}_t(t, x)$ as time tends to infinity.

The purpose of this paper is to show the unique existence theorem of solutions to the stationary problem corresponding to the problem (1.1)–(1.6) with time independent external magnetic field

[1] Given vector or matrix M, ${}^T M$ means the transposed M.

$\mathbf{H} = {}^T(H_1(x), H_2(x), H_3(x))$ near the constant state $\rho_0 = \alpha + i\beta$ with $|\rho_0|^2 = \alpha^2 + \beta^2 = 1$ and $(\Phi, \mathbf{A}) = (0, 0)$ in the $L_p(\Omega)$ framework and to show its stability concerning the initial disturbance by the small initial data in $L_3(\Omega)$. First, given $(\alpha, \beta) \in \mathbb{R}^2$ with $\alpha^2 + \beta^2 = 1$, by setting $\rho = \alpha + \xi + i(\beta + \zeta)$ and $\Psi = -\eta\Phi + \beta\xi - \alpha\zeta$ in (1.1)–(1.3), we see that real valued unknown functions ξ, ζ, $\mathbf{A}$ and Ψ enjoy the following equations:

$$\xi_t - \Delta\xi + c_1\xi + c_2\zeta = E_1(\xi, \zeta, \Psi) + F_1(\xi, \zeta, \mathbf{A}) \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.7}$$

$$\zeta_t - \Delta\zeta + c_2\xi + c_3\zeta = E_2(\xi, \zeta, \Psi) + F_2(\xi, \zeta, \mathbf{A}) \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.8}$$

$$\eta\mathbf{A}_t + \nabla \times (\nabla \times \mathbf{A}) + \mathbf{A} + \nabla \times \mathbf{H} + \eta\nabla\Psi = F_3(\xi, \zeta, \mathbf{A}) \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.9}$$

$$\nabla \cdot \mathbf{A} = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \tag{1.10}$$

$$\partial_\nu\xi|_\Gamma = \partial_\nu\zeta|_\Gamma = 0, \qquad \nu \cdot \mathbf{A}|_\Gamma = 0, \qquad (\nabla \times \mathbf{A} + \mathbf{H}) \times \nu|_\Gamma = 0, \tag{1.11}$$

$$(\xi, \zeta, \mathbf{A})|_{t=0} = (\xi_0, \zeta_0, \mathbf{A}_0) \tag{1.12}$$

where

$$\begin{aligned}
&c_1 = 2\kappa\alpha^2 + \eta^{-1}\beta^2, \qquad c_2 = \alpha\beta\big(2\kappa - \eta^{-1}\big), \qquad c_3 = 2\kappa\beta^2 + \eta^{-1}\alpha^2;\\
&E_1(\xi, \zeta, \Psi) = \eta^{-1}(\beta + \zeta)\Psi, \qquad E_2(\xi, \zeta, \Psi) = -\eta^{-1}(\alpha + \xi)\Psi;\\
&F_1(\xi, \zeta, \mathbf{A}) = 2(\mathbf{A} \cdot \nabla)\zeta - 3\kappa\alpha\xi^2 - \big(2\kappa + \eta^{-1}\big)\beta\xi\zeta + \big(\eta^{-1} - \kappa\big)\alpha\zeta^2\\
&\qquad - \alpha|A|^2 - \big(\kappa\big(\xi^2 + \zeta^2\big) + |A|^2\big)\xi;\\
&F_2(\xi, \zeta, \mathbf{A}) = -2(\mathbf{A} \cdot \nabla)\xi + \big(\eta^{-1} - \kappa\big)\beta\xi^2 - \big(2\kappa + \eta^{-1}\big)\alpha\xi\zeta - 3\kappa\beta\zeta^2\\
&\qquad - \beta|A|^2 - \big(\kappa\big(\xi^2 + \zeta^2\big) + |A|^2\big)\zeta;\\
&F_3(\xi, \zeta, \mathbf{A}) = \xi\nabla\zeta - \zeta\nabla\xi - \big(2\alpha\xi + \xi^2 + 2\beta\zeta + \zeta^2\big)\mathbf{A}.
\end{aligned} \tag{1.13}$$

Assuming that $\xi = \tilde{\xi}(x)$, $\zeta = \tilde{\zeta}(x)$, $\mathbf{A} = \tilde{\mathbf{A}}(x)$ and $\Psi = \tilde{\Psi}(x)$ in (1.7)–(1.12), we have the corresponding stationary problem:

$$-\Delta\tilde{\xi} + c_1\tilde{\xi} + c_2\tilde{\zeta} = E_1(\tilde{\xi}, \tilde{\zeta}, \tilde{\Psi}) + F_1(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \quad \text{in } \Omega, \tag{1.14}$$

$$-\Delta\tilde{\zeta} + c_2\tilde{\xi} + c_3\tilde{\zeta} = E_2(\tilde{\xi}, \tilde{\zeta}, \tilde{\Psi}) + F_2(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \quad \text{in } \Omega, \tag{1.15}$$

$$\nabla \times (\nabla \times \tilde{\mathbf{A}}) + \tilde{\mathbf{A}} + \nabla \times \tilde{\mathbf{H}} + \eta\nabla\tilde{\Psi} = F_3(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \quad \text{in } \Omega, \tag{1.16}$$

$$\nabla \cdot \tilde{\mathbf{A}} = 0 \quad \text{in } \Omega, \tag{1.17}$$

$$\partial_\nu\tilde{\xi}|_\Gamma = \partial_\nu\tilde{\zeta}|_\Gamma = 0, \qquad \nu \cdot \tilde{\mathbf{A}}|_\Gamma = 0, \qquad (\nabla \times \tilde{\mathbf{A}} + \tilde{\mathbf{H}}) \times \nu|_\Gamma = 0. \tag{1.18}$$

To treat Eqs. (1.9) and (1.10) with boundary condition (1.11) for $\mathbf{A}$ and Eqs. (1.16) and (1.17) with boundary condition (1.18) for $\tilde{\mathbf{A}}$, we introduce the Helmholtz decomposition, which was proved by Fujiwara and Morimoto [4] in the bounded domain case, by Miyakawa [8] and Simader and Sohr [10] in the exterior domain case and by Farwig and Sohr [3] in the perturbed half space case. Set

$$C_{0,\sigma}^{\infty}(\Omega) = \{u = {}^T(u_1, u_2, u_3) \in C_0^{\infty}(\Omega)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega\};$$
$$X_p(\Omega) = \text{the closure of } C_{0,\sigma}^{\infty}(\Omega) \text{ in } L_p(\Omega);$$
$$G_p(\Omega) = \{\nabla\pi \mid \pi \in \hat{W}_p^1(\Omega)^3\};$$
$$\hat{W}_p^1(\Omega) = \left\{\pi \in L_{p,\mathrm{loc}}(\overline{\Omega}) \,\middle|\, \nabla\pi \in L_p(\Omega)^3,\ \int_{\Omega_0} \pi\, dx = 0\right\},$$

where Ω_0 is a bounded domain contained in Ω. Then, for $1 < p < \infty$ there holds the Helmholtz decomposition:

$$L_p(\Omega)^3 = X_p(\Omega) \oplus G_p(\Omega) \quad \text{(direct sum)}. \tag{1.19}$$

It is well known that

$$X_p(\Omega) = \{u \in L_p(\Omega)^3 \mid \operatorname{div} u = 0 \text{ in } \Omega,\ \nu \cdot u|_\Gamma = 0\}. \tag{1.20}$$

It follows from the Helmholtz decomposition (1.19) that given $u \in L_p(\Omega)^3$, there exist unique $v \in X_p(\Omega)$ and $\pi \in \hat{W}_p^1(\Omega)$ such that $u = v + \nabla\pi$. Let us define the projection $P: L_p(\Omega) \to X_p(\Omega)$ by $Pu = v$, and then we know that

$$\|Pu\|_{L_p} \leqq C_p\|u\|_{L_p}, \tag{1.21}$$
$$\|\nabla\pi\|_{L_p} \leqq C_p\|u\|_{L_p}. \tag{1.22}$$

In order to state our main results, at this point we outline our notation. The norms in the Lebesgue space $L_p(\Omega)$ and in Sobolev space $W_p^k(\Omega)$, $k \geqq 1$, are denoted by $\|\cdot\|_{L_p}$ and $\|\cdot\|_{W_p^k}$, respectively. Given Banach spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively, we set

$$\|v\|_{X\cap Y} = \|v\|_X + \|v\|_Y,$$
$$X^n = \{v = {}^T(v_1, \ldots, v_n) \mid v_j \in X\}, \qquad \|v\|_X = \sum_{j=1}^n \|v_j\|_X.$$

We set

$$(u, v)_\Omega = \int_\Omega u(x)v(x)\,dx, \qquad (u, v)_\Gamma = \int_\Gamma u(x)v(x)\,d\sigma,$$

where $d\sigma$ is the surface element on Γ. $C = C_{a,b,\ldots}$ means that the constant C depends on the quantities $a, b, \ldots$ in the parenthesis. To denote generic constants we use the same letter C, and therefore the constants C and $C_{a,b,\ldots}$ may change from line to line.

Now, we introduce the notion of the weak solutions to (1.14)–(1.18).

Definition 1.1. Let p and p' be numbers such that $1 < p, p' < \infty$ and $1/p + 1/p' = 1$. Set

$$\dot{W}_p^1(\Omega) = X_p(\Omega) \cap W_p^1(\Omega)^3, \qquad Y_p(\Omega) = W_p^1(\Omega)^2 \times \dot{W}_p^1(\Omega).$$

Then, we say that $(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in Y_p(\Omega)$ is a weak solution to (1.14)–(1.18) if $(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}})$ satisfies the variational equation:

$$(\nabla\tilde{\xi}, \nabla\tilde{\xi}\,')_\Omega + c_1(\tilde{\xi}, \tilde{\xi}\,')_\Omega + c_2(\tilde{\zeta}, \tilde{\xi}\,')_\Omega = \big(E_1(\tilde{\xi}, \tilde{\zeta}, \tilde{\Psi}) + F_1(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}), \tilde{\xi}\,'\big)_\Omega; \tag{1.23}$$

$$(\nabla\tilde{\zeta}, \nabla\tilde{\zeta}\,')_\Omega + c_2(\tilde{\zeta}, \tilde{\zeta}\,')_\Omega + c_3(\tilde{\xi}, \tilde{\zeta}\,')_\Omega = \big(E_1(\tilde{\xi}, \tilde{\zeta}, \tilde{\Psi}) + F_2(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}), \tilde{\zeta}\,'\big)_\Omega; \tag{1.24}$$

$$(\nabla \times \tilde{\mathbf{A}}, \nabla \times \tilde{\mathbf{A}}\,')_\Omega + (\tilde{\mathbf{A}}, \tilde{\mathbf{A}}\,')_\Omega = \big(PF_3(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}), \tilde{\mathbf{A}}\,'\big)_\Omega - (H, \nabla \times \tilde{\mathbf{A}}\,')_\Omega \tag{1.25}$$

for any $(\tilde{\xi}', \tilde{\zeta}', \tilde{\mathbf{A}}') \in Y_{p'}(\Omega)$, where $\tilde{\Psi}$ is chosen in such a way that

$$F_3(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) = PF_3(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) + \eta\nabla\tilde{\Psi}. \tag{1.26}$$

Concerning the existence of stationary solutions, we have the following theorem.

Theorem 1.2. *There exists an $\epsilon > 0$ such that if $\mathbf{H} \in L_3(\Omega)^3 \cap L_6(\Omega)^3$ and*

$$\|\mathbf{H}\|_{L_3 \cap L_6} \leqq \epsilon, \tag{1.27}$$

then (1.14)–(1.18) *admits a weak solution $(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in Y_3(\Omega) \cap Y_6(\Omega)$ and $\tilde{\Psi} \in W_3^1(\Omega) \cap \hat{W}_{3/2}^1(\Omega)$, which satisfies the estimate*

$$\big\|(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}})\big\|_{W_3^1 \cap W_6^1} + \|\tilde{\Psi}\|_{W_3^1} + \|\nabla\tilde{\Psi}\|_{L_{3/2}} \leqq C\epsilon \tag{1.28}$$

with some constant $C > 0$ independent of ϵ.

In addition, the weak solution satisfying (1.28) *is unique.*

Now, we consider the stability of $(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}})$ concerning the initial disturbance. To do this, we set

$$\xi = \tilde{\xi} + \varphi, \qquad \zeta = \tilde{\zeta} + \psi, \qquad \mathbf{A} = \tilde{\mathbf{A}} + \mathbf{B}, \qquad \Psi = \tilde{\Psi} + \Xi \tag{1.29}$$

in (1.7)–(1.12), and then $(\varphi, \psi, \mathbf{B}, \Xi)$ enjoys the equations:

$$\varphi_t - \Delta\varphi + c_1\varphi + c_2\psi = \mathcal{E}_1(\varphi, \psi, \Xi) + \mathcal{F}_1(\varphi, \psi, \mathbf{B}) \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.30}$$

$$\psi_t - \Delta\psi + c_2\varphi + c_3\psi = \mathcal{E}_2(\varphi, \psi, \Xi) + \mathcal{F}_2(\varphi, \psi, \mathbf{B}) \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.31}$$

$$\mathbf{B}_t + \eta^{-1}(\nabla \times (\nabla \times \mathbf{B}) + \mathbf{B}) + \nabla\Xi = \eta^{-1}\mathcal{F}_3(\varphi, \psi, \mathbf{B}) \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.32}$$

$$\operatorname{div}\mathbf{B} = 0 \qquad \text{in } \Omega \times \mathbb{R}_+, \tag{1.33}$$

$$\partial_\nu\varphi|_\Gamma = \partial_\nu\psi|_\Gamma = 0, \qquad \nu \cdot \mathbf{B}|_\Gamma = 0, \qquad (\nabla \times \mathbf{B}) \times \nu|_\Gamma = 0, \tag{1.34}$$

$$(\varphi, \psi, \mathbf{B})|_{t=0} = (\varphi_0, \psi_0, \mathbf{B}_0) = (\xi_0 - \tilde{\xi}, \zeta - \tilde{\zeta}, \mathbf{A}_0 - \tilde{\mathbf{A}}), \tag{1.35}$$

where

$$\begin{aligned}
\mathcal{E}_1(\varphi,\psi,\Xi) &= \eta^{-1}(\beta+\tilde{\zeta})\Xi+\eta^{-1}\psi(\tilde{\Psi}+\Xi);\\
\mathcal{E}_2(\varphi,\psi,\Xi) &= -\eta^{-1}(\alpha+\tilde{\xi})\Xi-\eta^{-1}\varphi(\tilde{\Psi}+\Xi);\\
\mathcal{F}_1(\varphi,\psi,\mathbf{B}) &= 2\big(\tilde{\mathbf{A}}\cdot\nabla\psi+\mathbf{B}\cdot\nabla(\tilde{\zeta}+\psi)\big)-3\kappa\alpha\big(2\tilde{\xi}\varphi+\varphi^2\big)+\big(\eta^{-1}-\kappa\big)\alpha\big(2\tilde{\zeta}\psi+\psi^2\big)\\
&\quad-\big(2\kappa+\eta^{-1}\big)\beta(\tilde{\xi}\psi+\tilde{\zeta}\varphi+\varphi\psi)-\alpha\big(2\tilde{\mathbf{A}}\cdot\mathbf{B}+|\mathbf{B}|^2\big)\\
&\quad-\kappa\big(2\tilde{\xi}\varphi+2\tilde{\zeta}\psi+\varphi^2+\psi^2\big)\tilde{\xi}\\
&\quad-\big(2\tilde{\mathbf{A}}\cdot\mathbf{B}+|\mathbf{B}|^2\big)\tilde{\xi}-\big\{\kappa\big((\tilde{\xi}+\varphi)^2+(\zeta+\psi)^2\big)+|\tilde{\mathbf{A}}+\mathbf{B}|^2\big\}\varphi;\\
\mathcal{F}_2(\varphi,\psi,\mathbf{B}) &= -2\big(\tilde{\mathbf{A}}\cdot\nabla\varphi+\mathbf{B}\cdot\nabla(\tilde{\xi}+\varphi)\big)-3\kappa\beta\big(2\tilde{\zeta}\psi+\psi^2\big)+\big(\eta^{-1}-\kappa\big)\beta\big(2\tilde{\xi}\varphi+\varphi^2\big)\\
&\quad-\big(2\kappa+\eta^{-1}\big)\alpha(\tilde{\xi}\psi+\tilde{\zeta}\varphi+\varphi\psi)-\beta\big(2\tilde{\mathbf{A}}\cdot\mathbf{B}+|\mathbf{B}|^2\big)\\
&\quad-\kappa\big(2\tilde{\xi}\varphi+2\tilde{\zeta}\psi+\varphi^2+\psi^2\big)\tilde{\zeta}-\big(2\tilde{\mathbf{A}}\cdot\mathbf{B}+|\mathbf{B}|^2\big)\tilde{\zeta}\\
&\quad-\big\{\kappa\big((\tilde{\xi}+\varphi)^2+(\zeta+\psi)^2\big)+|\tilde{\mathbf{A}}+\mathbf{B}|^2\big\}\psi;\\
\mathcal{F}_3(\varphi,\psi,\mathbf{B}) &= \tilde{\xi}\nabla\psi+\varphi\nabla(\tilde{\zeta}+\psi)-\tilde{\zeta}\nabla\varphi-\psi\nabla(\tilde{\xi}+\varphi)-\big(2\alpha\varphi+2\tilde{\xi}\varphi+\varphi^2\big)\tilde{\mathbf{A}}\\
&\quad-\big(2\beta\psi+2\tilde{\zeta}\psi+\psi^2\big)\tilde{\mathbf{A}}\\
&\quad-\big(2\alpha(\tilde{\xi}+\varphi)+(\tilde{\xi}+\varphi)^2+2\beta(\tilde{\zeta}+\psi)+(\tilde{\zeta}+\psi)^2\big)\mathbf{B}.
\end{aligned}\tag{1.36}$$

To treat (1.30) and (1.31), we use the analytic semigroup $\{T_L(t)\}_{t\geqq 0}$ corresponding to the system of the heat equations:

$$\varphi_t-\Delta\varphi+c_1\varphi+c_2\psi=0 \quad \text{in } \Omega\times\mathbb{R}_+, \tag{1.37}$$

$$\psi_t-\Delta\psi+c_2\varphi+c_3\psi=0 \quad \text{in } \Omega\times\mathbb{R}_+, \tag{1.38}$$

$$\partial_\nu\varphi|_\Gamma=\partial_\nu\psi|_\Gamma=0, \qquad (\varphi,\psi)|_{t=0}=(\varphi_0,\psi_0). \tag{1.39}$$

The problem (1.37)–(1.39) and the corresponding resolvent equation are treated in Section 2, below. Applying the projection P to (1.32), we have

$$\mathbf{B}_t+\eta^{-1}(\nabla\times(\nabla\times\mathbf{B})+\mathbf{B})=\eta^{-1}P\mathcal{F}_3(\varphi,\psi,\mathbf{B}) \quad \text{in } \Omega\times\mathbb{R}_+, \tag{1.40}$$

$$\mathcal{F}_3(\varphi,\psi,\mathbf{B})=P\mathcal{F}_3(\varphi,\psi,\mathbf{B})+\eta\nabla\Xi. \tag{1.41}$$

Let us define the operator M by the formula

$$M\mathbf{B}=\eta^{-1}\big(\nabla\times(\nabla\times\mathbf{B})+\mathbf{B}\big) \quad \text{for } \mathbf{B}\in\mathcal{D}_p(M), \tag{1.42}$$

where the domain $\mathcal{D}_p(M)$ is defined by the relation

$$\mathcal{D}_p(M)=\big\{\mathbf{B}\in X_p(\Omega)\cap W_p^2(\Omega)^3 \;\big|\; (\nabla\times\mathbf{B})\times\nu|_\Gamma=0\big\}. \tag{1.43}$$

As we will see in Section 3, the operator M generates an analytic semigroup $\{T_M(t)\}_{t\geqq 0}$. By using $\{T_L(t)\}_{t\geqq 0}$ and $\{T_M(t)\}_{t\geqq 0}$, we reduce (1.30), (1.31) and (1.40) to the integral equation:

$$\begin{aligned}\begin{pmatrix}\varphi(t)\\ \psi(t)\end{pmatrix} &= T_L(t)\begin{pmatrix}\varphi_0\\ \psi_0\end{pmatrix}\\ &\quad+\int_0^t T_L(t-s)\begin{pmatrix}\mathcal{E}_1(\varphi(s),\psi(s),\Xi(s))+\mathcal{F}_1(\varphi(s),\psi(s),\mathbf{B}(s))\\ \mathcal{E}_2(\varphi(s),\psi(s),\Xi(s))+\mathcal{F}_2(\varphi(s),\psi(s),\mathbf{B}(s))\end{pmatrix}ds, \quad (1.44)\end{aligned}$$

$$\mathbf{B}(t)=T_M(t)\mathbf{B}_0+\eta^{-1}\int_0^t T_M(t-s)P\mathcal{F}_3\big(\varphi(s),\psi(s),\mathbf{B}(s)\big)\,ds, \tag{1.45}$$

$$\mathcal{F}_3\big(\varphi(s),\psi(s),\mathbf{B}(s)\big)=P\mathcal{F}_3\big(\varphi(s),\psi(s),\mathbf{B}(s)\big)+\eta\nabla\Xi(s). \tag{1.46}$$

Concerning the integral equations (1.44) and (1.45) with (1.46), we have the following theorem which shows the stability of the stationary solution $(\tilde{\xi},\tilde{\zeta},\tilde{\mathbf{A}},\tilde{\Psi})$ by the initial disturbance $(\varphi_0,\psi_0,\mathbf{B}_0)$. To state the theorem, we set

$$Z_p(\Omega)=L_p(\Omega)^2\times X_p(\Omega).$$

Theorem 1.3. *There exists a $\gamma>0$ such that for sufficiently small $\epsilon>0$, given any $\mathbf{H}\in L_3(\Omega)^3\cap L_6(\Omega)^3$ and initial data $(\varphi_0,\psi_0,\mathbf{B}_0)\in Z_3(\Omega)$ which satisfy the smallness assumption*:

$$\|\mathbf{H}\|_{L_3\cap L_6}+\big\|(\varphi_0,\psi_0,\mathbf{B}_0)\big\|_{L_3}\leqq\epsilon, \tag{1.47}$$

the integral equations (1.44) *and* (1.45) *with* (1.46) *admit a solution*

$$\begin{gathered}\big(\varphi(t),\psi(t),\mathbf{B}(t)\big)\in C\big([0,\infty),Z_3(\Omega)\big)\cap C\big((0,\infty),Z_6(\Omega)\cap Y_3(\Omega)\big),\\ \Xi(t)\in C\big((0,\infty),W_3^1(\Omega)\cap\hat{W}_{3/2}^1(\Omega)\big),\end{gathered}$$

which possess the properties:

$$\begin{aligned}&\big[(\varphi,\psi,\mathbf{B})\big]_{3,0,\gamma,t}+\big[(\varphi,\psi,\mathbf{B})\big]_{6,1/4,\gamma,t}+\big[(\varphi,\psi,\mathbf{B})\big]_{\infty,1/2,\gamma,t}+\big[\nabla(\varphi,\psi,\mathbf{B})\big]_{3,1/2,\gamma,t}\\ &\quad+[\Xi]_{3,1/2,\gamma,t}+[\nabla\Xi]_{3/2,1/2,\gamma,t}+[\nabla\Xi]_{3,1,\gamma,t}\leqq C\epsilon,\end{aligned} \tag{1.48}$$

$$\begin{gathered}\lim_{t\to0+}\big\|\big(\varphi(t),\psi(t),\mathbf{B}(t)\big)-(\varphi_0,\psi_0,\mathbf{B}_0)\big\|_{L_3}=0,\\ \lim_{t\to0+}\big[(\varphi,\psi,\mathbf{B})\big]_{6,1/4,\gamma,t}+\big[\nabla(\varphi,\psi,\mathbf{B})\big]_{3,1/2,\gamma,t}=0.\end{gathered} \tag{1.49}$$

In addition, the solution of the integral equations (1.44) *and* (1.45) *with* (1.46) *satisfying* (1.48) *and* (1.49) *is unique.*

Here and hereafter, we set

$$[\varphi]_{p,\ell,\gamma,t}=\sup_{0<s\leqq t}f_{\ell,\gamma}(s)\big\|\varphi(s)\big\|_{L_p}, \tag{1.50}$$

$$f_{\ell,\gamma}(s)=\begin{cases}s^\ell, & 0<s\leqq1,\\ e^{\gamma s}, & s\geqq1.\end{cases} \tag{1.51}$$

2. About the system of Laplace equations with Neumann boundary condition

In this section, we consider the equation

$$\vec{v}_t - \Delta\vec{v} + Q\vec{v} = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \qquad \partial_\nu \vec{v}|_\Gamma = 0, \qquad \vec{v}|_{t=0} = \vec{v}_0, \tag{2.1}$$

where $\vec{v} = {}^T(\varphi, \psi)$, $\vec{v}_0 = {}^T(\varphi_0, \psi_0)$ and

$$Q = \begin{bmatrix} c_1 & c_2 \\ c_2 & c_3 \end{bmatrix} = \begin{bmatrix} 2\kappa\alpha^2 + \eta^{-1}\beta^2 & \alpha\beta(2\kappa - \eta^{-1}) \\ \alpha\beta(2\kappa - \eta^{-1}) & 2\kappa\beta^2 + \eta^{-1}\alpha^2 \end{bmatrix}.$$

Q is a 2×2 symmetric matrix and its eigenvalues are two positive numbers 2κ and η^{-1}. Choose an orthogonal matrix R in such a way that

$${}^T R Q R = \begin{bmatrix} 2\kappa & 0 \\ 0 & \eta^{-1} \end{bmatrix}. \tag{2.2}$$

Multiplying (2.1) by ${}^T R$, we have

$$\begin{gathered} \left({}^T R\vec{v}\right)_t - \Delta\left({}^T R\vec{v}\right) + {}^T RQR\left({}^T R\vec{v}\right) = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \\ \partial_\nu\left({}^T R\vec{v}\right)|_\Gamma = 0, \qquad {}^T R\vec{v}|_{t=0} = {}^T R\vec{v}_0, \end{gathered}$$

and therefore setting ${}^T R\vec{v} = \vec{w} = {}^T(w_1, w_2)$ and ${}^T R\vec{v}_0 = \vec{w}_0 = {}^T(w_{01}, w_{02})$, we see that w_j, $j = 1, 2$, enjoy the equations

$$(w_j)_t - \Delta w_j + \ell_j w_j = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \qquad \partial_\nu w_j|_\Gamma = 0, \qquad w_j|_{t=0} = w_{0j} \tag{2.3}$$

for $j = 1, 2$ where $\ell_1 = 2\kappa$ and $\ell_2 = \eta^{-1}$. Equations (2.1) and (2.3) are equivalent. Therefore, it suffices to solve (2.3).

The following theorem is well known at least in the bounded domain case and it was proved by Akiyama et al. [2] under our assumption on the domain.

Theorem 2.1. *Let $1 < p < \infty$, $0 < \epsilon < \pi/2$ and $\delta > 0$. Then, for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$ and $f \in L_p(\Omega)$, there exists a unique solution $w \in W_p^2(\Omega)$ to the resolvent equation*

$$\lambda w - \Delta w = f \quad \text{in } \Omega, \qquad \partial_\nu w|_\Gamma = 0, \tag{2.4}$$

which satisfies the estimate

$$|\lambda| \|w\|_{L_p} + \|w\|_{W_p^2} \leqq C\|f\|_{L_p}, \tag{2.5}$$

$$|\lambda|^{1/2} \|w\|_{L_p} + \|w\|_{W_p^1} \leqq C\|f\|_{L_p} \tag{2.6}$$

for any $\lambda \in \Sigma_\epsilon$ with $|\lambda| \geqq \delta$, where we have set

$$\Sigma_\epsilon = \left\{\lambda \in \mathbb{C} \mid |\arg\lambda| < \pi - \epsilon\right\}.$$

Let us define the operator A_j by the formula: $A_j u = -\Delta u + \ell_j u$ for $u \in \mathcal{D}_p(L)$ where $\mathcal{D}_p(L)$ is defined by the formula

$$\mathcal{D}_p(L) = \left\{ u \in W_p^2(\Omega) \;\middle|\; \partial_\nu u|_\Gamma = 0 \right\}. \tag{2.7}$$

Then, by Theorem 2.1 the operator A_j generates an analytic semigroup $\{T_j(t)\}_{t \geqq 0}$ on $L_p(\Omega)$, which satisfies the estimate:

$$\begin{aligned} &\|T_j(t)a\|_{W_p^k} \leqq C_{p,\ell_j^*} e^{-\ell_j^* t} t^{-k/2} \|a\|_{L_p}, \quad \forall a \in L_p(\Omega),\ t > 0,\ k = 0, 2, \\ &\|T_j(t)a\|_{W_p^2} \leqq C_{p,\ell_j^*} e^{-\ell_j^* t} \|a\|_{W_p^2}, \quad \forall a \in \mathcal{D}_p(L),\ t > 0, \end{aligned} \tag{2.8}$$

where $\ell_j^* \in (0, \ell_j)$ and $W_p^0(\Omega) = L_p(\Omega)$. Therefore, if we define $\{T_L(t)\}_{t \geqq 0}$ by the formula

$$T_L(t)\vec{v}_0 = R \begin{bmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{bmatrix} {}^T R \vec{v}_0, \tag{2.9}$$

then $\vec{v}(t) = T_L(t)\vec{v}_0$ solves (2.1). From (2.8) it follows that

$$\left\| T_L(t)\vec{v}_0 \right\|_{W_p^k} \leqq C_p e^{-\delta t} t^{-k/2} \|\vec{v}_0\|_{L_p}, \quad \forall \vec{v}_0 \in L_p(\Omega)^2,\ t > 0,\ k = 0, 2, \tag{2.10}$$

$$\left\| T_L(t)\vec{v}_0 \right\|_{W_p^2} \leqq C_p e^{-\delta t} \|\vec{v}_0\|_{W_p^2}, \quad \forall \vec{v}_0 \in \mathcal{D}_p(L)^2,\ t > 0, \tag{2.11}$$

where $\delta = \min(2\kappa, \eta^{-1})/2$.

Now, we shall show the L_p–L_q estimate:

Theorem 2.2. *Let* $1 < p < \infty$ *and let* $\{T_L(t)\}_{t \geqq 0}$ *be an analytic semigroup defined by* (2.9). *Set*

$$\delta = \min\bigl(2\kappa, \eta^{-1}\bigr)/2.$$

Then, we have the following estimates:

$$\left\| T_L(t)\vec{v}_0 \right\|_{L_q} \leqq C_{p,q} e^{-\delta t} t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q})} \|\vec{v}_0\|_{L_p}, \quad \forall t > 0,\ \vec{v}_0 \in L_p(\Omega), \tag{2.12}$$

provided that $1 \leqq p \leqq q \leqq \infty$ $(p \neq \infty,\ q \neq 1)$;

$$\left\| \nabla T_L(t)\vec{v}_0 \right\|_{L_q} \leqq C_{p,q} e^{-\delta t} t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{2}} \|\vec{v}_0\|_{L_p}, \quad \forall t > 0,\ \vec{v}_0 \in L_p(\Omega), \tag{2.13}$$

provided that $1 < p \leqq q < \infty$.

Proof. Using the complex interpolation

$$W_p^s(\Omega) = \bigl(L_p(\Omega), W_p^2(\Omega)\bigr)_\theta, \qquad 2\theta = s,$$

by (2.10) we have

$$\|T_L(t)\vec{v}_0\|_{W_p^s} \leqq C_{s,p} e^{-\delta t} t^{-s/2} \|\vec{v}_0\|_{L_p}. \tag{2.14}$$

Set $s = 3(1/p - 1/q)$. Assume first that $0 < s < 2$. Since $W_p^s(\Omega) \subset L_q(\Omega)$, it follows from (2.14) that

$$\|T_L(t)\vec{v}_0\|_{L_q} \leqq C_s \|T_L(t)\vec{v}_0\|_{W_p^s} \leqq C_{p,q} e^{-\delta t} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|\vec{v}_0\|_{L_p} \tag{2.15}$$

for any $\vec{v}_0 \in L_p(\Omega)^3$ with $3(1/p - 1/q) \in (0, 2)$. Assuming next that $s \geqq 2$, we choose $p_1, \ldots, p_{m-1}$ in such a way that $p = p_0 < p_1 < \cdots < p_{m-1} < p_m = q$ and $0 < 3(1/p_\ell - 1/p_{\ell+1}) < 2$ for $\ell = 0, 1, 2, \ldots, m-1$. Then, by (2.15) we have

$$\begin{aligned}
\|T_L(t)\vec{v}_0\|_{L_q} &= \|T_L\big(\underbrace{(t/m) + \cdots + (t/m)}_{m \text{ times}}\big)\vec{v}_0\|_{L_q} \\
&\leqq C e^{-\delta t/m} t^{-\frac{3}{2}(\frac{1}{p_m}-\frac{1}{p_{m-1}})} \|T_L\big(\underbrace{(t/m) + \cdots + (t/m)}_{m-1 \text{ times}}\big)\vec{v}_0\|_{L_{p_{m-1}}} \\
&\leqq C_{p,q} e^{-\delta t} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|\vec{v}_0\|_{L_p},
\end{aligned}$$

which shows (2.12) provided that $1 < p \leqq q < \infty$. Combining (2.14) with $s = 1$ and (2.15), we have

$$\begin{aligned}
\|\nabla T_L(t)\vec{v}_0\|_{L_q} &= \|\nabla T_L\big((t/2) + (t/2)\big)\vec{v}_0\|_{L_q} \leqq C e^{-\delta t/2} t^{-1/2} \|T_L(t/2)\vec{v}_0\|_{L_q} \\
&\leqq C_{p,q} e^{-\delta t} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|\vec{v}_0\|_{L_p}
\end{aligned}$$

provided that $1 < p \leqq q < \infty$. By the Gagliardo–Nirenberg inequality, we have for $r > \max(3, p)$,

$$\begin{aligned}
\|T_L(t)\vec{v}_0\|_{L_\infty} &\leqq C \|T_L(t)\vec{v}_0\|_{W_r^1}^{3/r} \|T_L(t)\vec{v}_0\|_{L_r}^{1-(3/r)} \\
&\leqq C\big[e^{-\delta t} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} \|\vec{v}_0\|_{L_p}\big]^{3/r} \big[e^{-\delta t} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{r})} \|\vec{v}_0\|_{L_p}\big]^{1-(3/r)} \\
&= C e^{-\delta t} t^{-\frac{3}{2p}} \|\vec{v}_0\|_{L_p}
\end{aligned}$$

provided that $1 < p < \infty$. Finally, the duality argument and the density of $C_0^\infty(\Omega)^n$ in $L_1(\Omega)^n$ implies that

$$\|T_L(t)\vec{v}_0\|_{L_p} \leqq C e^{-\delta t} t^{-\frac{3}{2}(1-\frac{1}{p})} \|\vec{v}_0\|_{L_1},$$

which completes the proof of the theorem. □

Let p and p' be numbers such that $1 < p,\ p' < \infty$ and $1/p + 1/p' = 1$. If $(\xi, \zeta) \in W_p^1(\Omega)^2$ satisfies the variational equations

$$
\begin{aligned}
(\nabla\xi, \nabla\xi')_\Omega + c_1(\xi, \xi')_\Omega + c_2(\zeta, \xi')_\Omega &= (f, \xi')_\Omega, \\
(\nabla\zeta, \nabla\zeta')_\Omega + c_2(\xi, \zeta')_\Omega + c_3(\zeta, \zeta')_\Omega &= (g, \zeta')_\Omega
\end{aligned} \tag{2.16}
$$

for any $(\xi', \zeta') \in W^1_{p'}(\Omega)^2$, we say that (ξ, ζ) is a weak solution to the system of equations

$$
\begin{gathered}
-\Delta\xi + c_1\xi + c_2\zeta = f, \qquad -\Delta\zeta + c_2\xi + c_3\zeta = g \quad \text{in } \Omega, \\
\partial_\nu\xi|_\Gamma = \partial_\nu\zeta|_\Gamma = 0.
\end{gathered} \tag{2.17}
$$

If $(\xi, \zeta) \in W^2_p(\Omega)^2$ solves (2.17), then integration by parts implies that ξ and ζ satisfy (2.16). By using the orthogonal transform R, (2.17) is reduced to the two equations

$$
-\Delta w_j + \ell_j w_j = f_j \quad \text{in } \Omega, \qquad \partial_\nu w_j|_\Gamma = 0
$$

for $j = 1, 2$ where $\ell_1 = 2\kappa$ and $\ell_2 = \eta^{-1}$. Therefore, by Theorem 2.1 with $\lambda = 2\kappa$ and $\lambda = \eta^{-1}$ and the density argument, we have the following theorem.

Theorem 2.3. *Let* $1 < p < \infty$. *Then, for every* $(f, g) \in L_p(\Omega)^2$ (2.17) *admits a unique weak solution* $(\xi, \zeta) \in W^1_p(\Omega)^2$ *which satisfies the estimate*

$$
\|(\xi, \zeta)\|_{W^1_p} \leqq C\|(f, g)\|_{L_p}. \tag{2.18}
$$

3. About the parabolic Maxwell equation

In this section, we consider the equations

$$
\begin{gathered}
\mathbf{B}_t + \eta^{-1}\big[\nabla \times (\nabla \times \mathbf{B}) + \mathbf{B}\big] = 0, \qquad \nabla \cdot \mathbf{B} = 0 \quad \text{in } \Omega \times \mathbb{R}_+, \\
\nu \cdot \mathbf{B}|_\Gamma = 0, \qquad (\nabla \times \mathbf{B}) \times \nu|_\Gamma = 0, \qquad \mathbf{B}|_{t=0} = \mathbf{B}_0,
\end{gathered} \tag{3.1}
$$

where $\mathbf{B} = {}^T(B_1, B_2, B_3)$ and $\mathbf{B}_0 = {}^T(B_{01}, B_{02}, B_{03})$. Let us consider the resolvent problem corresponding to (3.1):

$$
\begin{gathered}
\lambda\mathbf{A} + \eta^{-1}\big[\nabla \times (\nabla \times \mathbf{A})\big] + \nabla\Psi + \nabla \times \mathbf{H} = \mathbf{F}, \qquad \nabla \cdot \mathbf{A} = 0 \quad \text{in } \Omega, \\
\nu \cdot \mathbf{A}|_\Gamma = 0, \qquad (\nabla \times \mathbf{A} + \mathbf{H}) \times \nu|_\Gamma = 0.
\end{gathered} \tag{3.2}
$$

From Akiyama et al. [2] we know the following theorem concerning (3.2).

Theorem 3.1. *Let* $1 < p < \infty$, $0 < \epsilon < \pi/2$ *and* $\delta > 0$. *Let* P *be the projection corresponding to the Helmholtz decomposition* (1.19). *If we choose* $\Psi \in \hat{W}^1_p(\Omega)$ *in such a way that*

$$
\mathbf{F} = P\mathbf{F} + \nabla\Psi, \tag{3.3}
$$

then for every $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, $\mathbf{H} \in W^1_p(\Omega)^3$ *and* $\mathbf{F} \in L_p(\Omega)^3$, (3.2) *admits a unique solution* $\mathbf{A} \in W^2_p(\Omega)^3$, *which satisfies the estimate*

$$|\lambda|\|\mathbf{A}\|_{L_p} + \|\mathbf{A}\|_{W_p^2} \leqq C\big\{\|\mathbf{F}\|_{L_p} + |\lambda|^{1/2}\|\mathbf{H}\|_{L_p} + \|\nabla\mathbf{H}\|_{L_p}\big\}, \tag{3.4}$$

$$|\lambda|^{1/2}\|\mathbf{A}\|_{L_p} + \|\mathbf{A}\|_{W_p^1} \leqq C\big\{\|\mathbf{F}\|_{L_p} + \|\mathbf{H}\|_{L_p}\big\} \tag{3.5}$$

for any $\lambda \in \Sigma_\epsilon$ *with* $|\lambda| \geqq \delta$ *with some constant* $C = C_{p,\epsilon,\delta}$.

Let M be a linear operator defined by (1.42) with domain $\mathcal{D}_p(M)$ defined by (1.43). Then, by Theorem 3.1, M generates an analytic semigroup $\{T_M(t)\}_{t\geqq 0}$ on $X_p(\Omega)$. Moreover, we have

$$\begin{aligned}
&\big\|T_M(t)\mathbf{B}_0\big\|_{W_p^k} \leqq C_p e^{-t/2} t^{-k/2}\|\mathbf{B}_0\|_{L_p}, \quad \forall t>0,\ \mathbf{B}_0 \in X_p(\Omega),\ k=0,2,\\
&\big\|T_M(t)\mathbf{B}_0\big\|_{W_p^2} \leqq C_p e^{-t/2}\|\mathbf{B}_0\|_{W_p^2}, \quad \forall t>0,\ \mathbf{B}_0 \in \mathcal{D}_p(M).
\end{aligned} \tag{3.6}$$

Employing the same argument as in Section 2, from (3.6) we have the following theorem.

Theorem 3.2. *The analytic semigroup* $\{T_M(t)\}_{t\geqq 0}$ *satisfies the following estimates*:

$$\big\|T_M(t)\mathbf{B}_0\big\|_{L_q} \leqq C_{p,q} e^{-t/2} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})}\|\mathbf{B}_0\|_{L_p}, \quad \forall t>0,\ \mathbf{B}_0 \in L_p(\Omega)^3 \tag{3.7}$$

provided that $1 \leqq p \leqq q \leqq \infty$ $(p \neq \infty, q \neq 1)$;

$$\big\|\nabla T_M(t)\mathbf{B}_0\big\|_{L_q} \leqq C_{p,q} e^{-t/2} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}}\|\mathbf{B}_0\|_{L_p}, \quad \forall t>0,\ \mathbf{B}_0 \in L_p(\Omega)^3 \tag{3.8}$$

provided that $1 < p \leqq q < \infty$.

Finally, we consider the weak solution to the equations

$$\begin{aligned}
&\nabla\times(\nabla\times\mathbf{A}) + \mathbf{A} + \nabla\times\mathbf{H} + \eta\nabla\Psi = \mathbf{F} \qquad \text{in } \Omega,\\
&\nabla\cdot\mathbf{A} = 0 \qquad \text{in } \Omega,\\
&\nu\cdot\mathbf{A}|_\Gamma = 0, \qquad (\nabla\times\mathbf{A}+\mathbf{H})\times\nu|_\Gamma = 0.
\end{aligned} \tag{3.9}$$

Let p and p' be numbers such that $1 < p,\ p' < \infty$ and $1/p + 1/p' = 1$. If $\mathbf{A} \in \dot{W}_p^1(\Omega)^3$ satisfies the variational equation

$$(\nabla\times\mathbf{A}, \nabla\times\mathbf{A}')_\Omega + (\mathbf{A}, \mathbf{A}')_\Omega = (P\mathbf{F}, \mathbf{A}')_\Omega - (\mathbf{H}, \nabla\times\mathbf{A}')_\Omega \tag{3.10}$$

for any $\mathbf{A}' \in \dot{W}_{p'}^1(\Omega)^3$, then we say that $\mathbf{A} \in \dot{W}_p^1(\Omega)^3$ is a weak solution to (3.9). If $\mathbf{A} \in W_p^2(\Omega)^3$ satisfies (3.9), then by using the formula

$$(\nabla\times\mathbf{A}, \mathbf{A}')_\Omega = -(\mathbf{A}\times\nu, \mathbf{A}')_\Gamma + (\mathbf{A}, \nabla\times\mathbf{A}')_\Omega, \tag{3.11}$$

we see that $\mathbf{A}$ also satisfies (3.10). Therefore, using Theorem 3.1 and the density argument, we have the following theorem.

Theorem 3.3. *Let p and p' be numbers such that $1 < p,\ p' < \infty$ and $1/p + 1/p' = 1$. Then, for any $\mathbf{H} \in L_p(\Omega)^3$ and $\mathbf{F} \in L_p(\Omega)^3$ (3.9) admits a unique weak solution $\mathbf{A} \in \dot{W}_p^1(\Omega)^3$ which satisfies the estimate*

$$\|\mathbf{A}\|_{W_p^1} \leqq C\big(\|\mathbf{F}\|_{L_p} + \|\mathbf{H}\|_{L_p}\big). \tag{3.12}$$

4. A proof of Theorem 1.2

In this section, we shall show Theorem 1.2 by the contraction mapping principle. To do this, as the underlying space we set

$$\mathbb{I}_\epsilon = \big\{\tilde{U} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in Y_3(\Omega) \cap Y_6(\Omega) \bigm| \|\tilde{U}\|_{W_3^1 \cap W_6^1} \leqq \epsilon\big\}, \tag{4.1}$$

where ϵ is a positive number which is smaller than 1. It will be determined later. Below, to estimate the nonlinear terms, we shall use the following Sobolev's inequalities:

$$\|u\|_{L_\infty} \leqq C\|u\|_{W_6^1}, \qquad \|u\|_{L_6} \leqq C\|u\|_{W_3^1} \tag{4.2}$$

and the following estimates:

$$\begin{aligned}
&\|u\nabla v\|_{L_{3/2}} \leqq \|u\|_{L_3}\|\nabla v\|_{L_3}, \qquad \|uv\|_{L_{3/2}} \leqq \|u\|_{L_3}\|v\|_{L_3},\\
&\|uvw\|_{L_{3/2}} \leqq \|u\|_{L_3}\|v\|_{L_6}\|w\|_{L_6},\\
&\|u\nabla v\|_{L_3} \leqq \|u\|_{L_\infty}\|\nabla v\|_{L_3} \leqq C\|u\|_{W_6^1}\|\nabla v\|_{L_3},\\
&\|uv\|_{L_3} \leqq \|u\|_{L_\infty}\|v\|_{L_3} \leqq C\|u\|_{W_6^1}\|v\|_{L_3},\\
&\|uvw\|_{L_3} \leqq \|u\|_{L_\infty}\|v\|_{L_6}\|w\|_{L_6} \leqq C\|u\|_{W_6^1}\|v\|_{L_6}\|w\|_{L_6},\\
&\|u\nabla v\|_{L_6} \leqq \|u\|_{L_\infty}\|\nabla v\|_{L_6} \leqq C\|u\|_{W_6^1}\|\nabla v\|_{L_6},\\
&\|uv\|_{L_6} \leqq \|u\|_{L_\infty}\|v\|_{L_6} \leqq C\|u\|_{W_6^1}\|v\|_{L_6},\\
&\|uvw\|_{L_6} \leqq \|u\|_{L_\infty}\|v\|_{L_\infty}\|w\|_{L_6} \leqq C\|u\|_{W_6^1}\|v\|_{W_6^1}\|w\|_{L_6}.
\end{aligned} \tag{4.3}$$

First, we consider the Helmholtz decomposition of the nonlinear term $F_3(\tilde{U})$. If $\Psi \in \hat{W}_p^1(\Omega)$ and $1 < p < 3$, then we have

$$\|\Psi\|_{L_{3p/(3-p)}} \leqq C_p\|\nabla\Psi\|_{L_p}, \tag{4.4}$$

which is proved in Appendix A. Given $\tilde{U} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in \mathbb{I}_\epsilon$ we choose $\tilde{\Psi}_{\tilde{U}}$ in such a way that

$$F_3(\tilde{U}) = PF_3(\tilde{U}) + \eta\nabla\tilde{\Psi}_{\tilde{U}}. \tag{4.5}$$

Applying (1.22) with $p = 3$ and $3/2$ to (4.5) and using (4.4) with $p = 3/2$ and the Sobolev's inequality (4.2), we have

$$\|\tilde{\Psi}_{\tilde{U}}\|_{L_6} + \|\tilde{\Psi}_{\tilde{U}}\|_{W_3^1} + \|\nabla\tilde{\Psi}_{\tilde{U}}\|_{L_{3/2}} \leqq C\big\|F_3(\tilde{U})\big\|_{L_{3/2}\cap L_3}. \tag{4.6}$$

By (4.3), we have

$$\left\| F_3(\tilde{U}) \right\|_{L_{3/2}} \leqq C\epsilon^2, \qquad \left\| F_3(\tilde{U}) \right\|_{L_3} \leqq C\epsilon^2; \tag{4.7}$$

provided that $\|(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}})\|_{W_3^1 \cap W_6^1} \leqq \epsilon$. Here and hereafter, we use the inequality: $\epsilon^k \leqq \epsilon$ for $k \geqq 2$, which follows from the fact: $0 < \epsilon \leqq 1$. Therefore, combining (4.6) and (4.7) implies that

$$\|\tilde{\Psi}_{\tilde{U}}\|_{L_6} + \|\tilde{\Psi}_{\tilde{U}}\|_{W_3^1} + \|\nabla \tilde{\Psi}_{\tilde{U}}\|_{L_{3/2}} \leqq C\epsilon^2, \tag{4.8}$$

provided that $\tilde{U} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in \mathbb{I}_\epsilon$. By (4.7), (4.8) and (4.3), we have

$$\left\| E_j(\tilde{\xi}, \tilde{\zeta}, \tilde{\Psi}_{\tilde{U}}) \right\|_{L_3 \cap L_6} \leqq C\epsilon^2 \quad (j = 1, 2), \qquad \left\| F_j(\tilde{U}) \right\|_{L_3 \cap L_6} \leqq C\epsilon^2 \quad (j = 1, 2, 3) \tag{4.9}$$

provided that $\tilde{U} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in \mathbb{I}_\epsilon$.

In the same manner, we have

$$\begin{gathered}
\|\tilde{\Psi}_{\tilde{U}_1} - \tilde{\Psi}_{\tilde{U}_2}\|_{L_6} + \|\tilde{\Psi}_{\tilde{U}_1} - \tilde{\Psi}_{\tilde{U}_2}\|_{W_3^1} + \left\| \nabla(\tilde{\Psi}_{\tilde{U}_1} - \tilde{\Psi}_{\tilde{U}_2}) \right\|_{L_{3/2}} \leqq C\epsilon \|\tilde{U}_1 - \tilde{U}_2\|_{W_3^1 \cap W_6^1}, \\
\sum_{k=1}^{3} \left\| F_k(\tilde{U}_1) - F_k(\tilde{U}_2) \right\|_{L_3 \cap L_6} \leqq C\epsilon \|\tilde{U}_1 - \tilde{U}_2\|_{W_3^1 \cap W_6^1}, \\
\sum_{k=1}^{2} \left\| E_k(\tilde{\xi}_1, \tilde{\zeta}_1, \tilde{\Psi}_{\tilde{U}_1}) - E_k(\tilde{\xi}_2, \tilde{\zeta}_2, \tilde{\Psi}_{\tilde{U}_2}) \right\|_{L_3 \cap L_6} \leqq C\epsilon \|\tilde{U}_1 - \tilde{U}_2\|_{W_3^1 \cap W_6^1}
\end{gathered} \tag{4.10}$$

provided that $\tilde{U}_j = (\tilde{\xi}_j, \tilde{\zeta}_j, \tilde{\mathbf{A}}_j) \in \mathbb{I}_\epsilon$, $j = 1, 2$. By Theorems 2.3 and 3.3, there exists a unique $(\xi, \zeta, \mathbf{A}) \in Y_3(\Omega) \cap Y_6(\Omega)$, which solves the equations

$$\begin{aligned}
(\nabla\xi, \nabla\tilde{\xi}')_\Omega + c_1(\xi, \tilde{\xi}')_\Omega + c_2(\zeta, \tilde{\xi}')_\Omega &= \left(E_1(\tilde{\xi}, \tilde{\zeta}, \tilde{\Psi}_{\tilde{U}}) + F_1(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}), \tilde{\xi}'\right)_\Omega, \\
(\nabla\zeta, \nabla\tilde{\zeta}')_\Omega + c_2(\xi, \tilde{\zeta}')_\Omega + c_3(\zeta, \tilde{\zeta}')_\Omega &= \left(E_2(\tilde{\xi}, \tilde{\zeta}, \tilde{\Psi}_{\tilde{U}}) + F_2(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}), \tilde{\zeta}'\right)_\Omega, \\
(\nabla \times \mathbf{A}, \nabla \times \tilde{\mathbf{A}}')_\Omega + (\mathbf{A}, \tilde{\mathbf{A}}')_\Omega &= \left(PF_3(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}), \tilde{\mathbf{A}}'\right)_\Omega - (\mathbf{H}, \nabla \times \tilde{\mathbf{A}}')_\Omega
\end{aligned} \tag{4.11}$$

for any $(\tilde{\xi}', \tilde{\zeta}', \tilde{\mathbf{A}}') \in Y_{p'}(\Omega)$ with $p' = 3/2$ and $p' = 6/5$. Here we note that $p' = 3/2$ and $p' = 6/5$ are conjugate exponents corresponding to $p = 3$ and $p = 6$, respectively. And also, by Theorems 2.3 and 3.3 and (4.9), we have

$$\left\| (\xi, \zeta, \mathbf{A}) \right\|_{W_3^1 \cap W_6^1} \leqq C\epsilon^2 + C\|\mathbf{H}\|_{L_3 \cap L_6}. \tag{4.12}$$

Therefore, we define the map K by $(\xi, \zeta, \mathbf{A}) = K(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}})$. Then, assuming that

$$C\epsilon \leqq 1/2, \qquad C\|\mathbf{H}\|_{L_3 \cap L_6} \leqq \epsilon/2, \tag{4.13}$$

by (4.12) we have

$$\left\| K(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \right\|_{W_3^1 \cap W_6^1} \leqq \epsilon \tag{4.14}$$

provided that $\tilde{U} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in \mathbb{I}_\epsilon$, which implies that K is a map from $\mathbb{I}_\epsilon$ into itself.

Given $\tilde{U}_j = (\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in \mathbb{U}_\epsilon$, $j = 1, 2$, by Theorems 2.3 and 3.3 and (4.10) we have

$$\|K\tilde{U}_1 - K\tilde{U}_2\|_{W_3^1 \cap W_6^1} \leqq C\epsilon \|\tilde{U}_1 - \tilde{U}_2\|_{W_3^1 \cap W_6^1}. \tag{4.15}$$

Since $C\epsilon \leqq 1/2$ as follows from (4.13), (4.15) implies that K is a contraction map of $\mathbb{I}_\epsilon$, and therefore the map K has a unique fixed point $\tilde{U} = (\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in \mathbb{I}_\epsilon$, which solves (1.23)–(1.25). Since $\tilde{\Psi}_{\tilde{U}}$ is defined by (4.5), we have (4.8). Therefore, we have proved the existence of a weak solution $(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}) \in Y_3 \cap Y_6$ to (1.14)–(1.18) which possesses the estimate (1.28). This completes the proof of Theorem 1.2.

5. A proof of Theorem 1.3

In this section we shall prove Theorem 1.3 by contraction mapping principle. For the notational simplicity, we set

$$\begin{pmatrix} \varphi(t) \\ \psi(t) \\ \mathbf{B}(t) \end{pmatrix}, \qquad T(t)U_0 = \begin{pmatrix} T_L(t)\begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} \\ T_M(t)\mathbf{B}_0 \end{pmatrix}, \qquad U_0 = \begin{pmatrix} \varphi_0 \\ \psi_0 \\ \mathbf{B}_0 \end{pmatrix}.$$

Setting $2\gamma = \min(\delta, 1/2)$, by Theorems 2.2 and 3.2 we have

$$\left\|T(t)U_0\right\|_{L_q} \leqq C_{p,q} e^{-2\gamma t} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})} \|U_0\|_{L_p} \tag{5.1}$$

for any $U_0 \in Z_p(\Omega)$ and $t > 0$ provided that $1 \leqq p \leqq q \leqq \infty$ $(p \neq \infty, q \neq 1)$ and

$$\left\|\nabla T(t)U_0\right\|_{L_q} \leqq C_{p,q} e^{-2\gamma t} t^{-\frac{3}{2}(\frac{1}{p}-\frac{1}{q})-\frac{1}{2}} \|U_0\|_{L_p} \tag{5.2}$$

for any $U_0 \in Z_p(\Omega)$ and $t > 0$ provided that $1 < p \leqq q < \infty$. Moreover, by (2.11) and (3.6) we have

$$\left\|T(t)U_0\right\|_{W_p^2} + \left\|\partial_t T(t)U_0\right\|_{L_p} \leqq C_p e^{-2\gamma t} \|U_0\|_{W_p^2} \tag{5.3}$$

for any $U_0 \in \mathcal{D}_p(L)^2 \times \mathcal{D}_p(M)$.

By using $T(t)$, (1.44) and (1.45) are rewritten in the form

$$U(t) = T(t)U_0 + \sum_{j=1}^{3} \int_0^t T(t-s) G_j\big(U(s)\big)\,ds, \tag{5.4}$$

where

$$G_1\big(U(s)\big) = {}^T\big(\eta^{-1}\beta\, \Xi_U(s), -\eta^{-1}\alpha\, \Xi_U(s), 0\big)$$
$$G_2\big(U(s)\big) = \begin{pmatrix} \eta^{-1}\tilde{\zeta}\, \Xi_U(s) + \eta^{-1}\psi(s)(\tilde{\Psi} + \Xi_U(s)) \\ -\eta^{-1}\tilde{\xi}\, \Xi_U(s) - \eta^{-1}\varphi(s)(\tilde{\Psi} + \Xi_U(s)) \\ 0 \end{pmatrix},$$

$$G_3\big(U(s)\big) = {}^T\big(\mathcal{F}_1\big(U(s)\big), \mathcal{F}_2\big(U(s)\big), \mathcal{F}_3\big(U(s)\big)\big)$$

and $\Xi_U(s)$ is chosen in such a way that

$$\mathcal{F}_3\big(U(s)\big) = P\mathcal{F}_3\big(U(s)\big) + \eta\nabla\Xi_U(s). \tag{5.5}$$

Let ϵ be a small positive number determined later. To solve the integral equation (5.4), as the underlying space we set

$$\begin{aligned}
\mathbb{J}_\epsilon = \Big\{ & U(t) \in C\big([0,\infty), Z_3(\Omega)\big) \cap C\big((0,\infty), Z_6(\Omega) \cap Y_3(\Omega)\big) \ \Big| \\
& |||U|||_t \leqq \epsilon \ \forall t > 0, && (5.6)\\
& \lim_{t\to 0+} \big\|U(t) - U_0\big\|_{L_3} = 0,\ \lim_{t\to 0+} \big([U]_{6,1/4,\gamma,t} + [\nabla U]_{3,1/2,\gamma,t}\big) = 0\Big\}, && (5.7)
\end{aligned}$$

where we have set

$$|||U|||_t = [U]_{3,0,\gamma,t} + [U]_{6,1/4,\gamma,t} + [\nabla U]_{3,1/2,\gamma,t}.$$

Set

$$\Phi U(t) = T(t)U_0 + \Phi_1 U(t) + \Phi_2 U(t),$$

$$\Phi_1 U(t) = \int_0^t T(t-s)G_1\big(U(s)\big)\,ds, \qquad \Phi_2 U(t) = \sum_{j=2}^3 \int_0^t T(t-s)G_j\big(U(s)\big)\,ds.$$

We start with the following lemma.

Lemma 5.1. *Given $U_0 \in Z_3(\Omega)$, we have*

$$\lim_{t\to 0+} \big\|T(t)U_0 - U_0\big\|_{L_3} = 0, \tag{5.8}$$

$$\lim_{t\to 0+} \big(\big[T(\cdot)U_0\big]_{6,1/4,\gamma,t} + \big[\nabla T(\cdot)U_0\big]_{3,1/2,\gamma,t}\big) = 0, \tag{5.9}$$

$$\big|\big|\big|T(\cdot)U_0\big|\big|\big|_t \leqq C\|U_0\|_{L_3}. \tag{5.10}$$

Proof. Since $C_0^\infty(\Omega)^2 \times C_{0,\sigma}^\infty(\Omega)$ is dense in $L_3(\Omega)^2 \times X_3(\Omega)$, given any $\delta > 0$ there exists a $\tilde{U}_0 \in C_0^\infty(\Omega)^2 \times C_{0,\sigma}^\infty(\Omega)$ such that $\|U_0 - \tilde{U}_0\|_{L_3} < \delta$. To obtain

$$\overline{\lim_{t\to 0+}} \big\|T(t)U_0 - U_0\big\|_{L_3} \leqq \delta,$$

by (5.1) and (5.3) we observe that

$$\begin{aligned}\|T(t)U_0 - U_0\|_{L_3} &\leqq \|T(t)(U_0 - \tilde{U}_0)\|_{L_3} + \|T(t)\tilde{U}_0 - \tilde{U}_0\|_{L_3} + \|\tilde{U}_0 - U_0\|_{L_3} \\ &\leqq \left(C_{3,3}e^{-2\gamma t} + 1\right)\delta + \int_0^t \|\partial_s T(s)\tilde{U}_0\|_{L_3}\,ds \\ &\leqq (C_{3,3}+1)\delta + C_{3,3}t\|\tilde{U}_0\|_{W_3^2}.\end{aligned}$$

The arbitrariness of choice of δ implies (5.8).

To obtain

$$\overline{\lim_{t\to 0+}}\left[T(\cdot)U_0\right]_{6,1/4,\gamma,t} \leqq C_{3,6}\delta,$$

by (5.1) we observe that

$$\begin{aligned}\|T(t)U_0\|_{L_6} &\leqq \|T(t)(U_0 - \tilde{U}_0)\|_{L_6} + \|T(t)\tilde{U}_0\|_{L_6} \\ &\leqq C_{3,6}t^{-1/4}e^{-2\gamma t}\delta + C_{6,6}e^{-2\gamma t}\|\tilde{U}_0\|_{L_6}.\end{aligned}$$

The arbitrariness of choice of δ implies that

$$\lim_{t\to 0+}\left[T(\cdot)U_0\right]_{6,1/4,\gamma,t} = 0.$$

To obtain

$$\overline{\lim_{t\to 0+}}\left[\nabla T(\cdot)U_0\right]_{3,1/2,\gamma,t} \leqq C_{3,6}\delta,$$

by (5.2) we observe that

$$\begin{aligned}\|\nabla T(t)U_0\|_{L_3} &\leqq \|\nabla T(t)(U_0 - \tilde{U}_0)\|_{L_3} + \|\nabla T(t)\tilde{U}_0\|_{L_3} \\ &\leqq C_{3,3}t^{-1/2}e^{-2\gamma t}\delta + C_{3,3}e^{-2\gamma t}\|\tilde{U}_0\|_{W_3^2}.\end{aligned}$$

The arbitrariness of choice of δ implies that

$$\lim_{t\to 0+}\left[\nabla T(\cdot)U_0\right]_{3,1/2,\gamma,t} = 0.$$

Therefore, we have (5.9).

By (5.1) and (5.2) we have (5.10), which completes the proof of the lemma. □

Assume that the stationary solution $(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}}, \tilde{\Psi})$ obtained by Theorem 1.2 satisfies the estimate

$$\|(\tilde{\xi}, \tilde{\zeta}, \tilde{\mathbf{A}})\|_{W_3^1\cap W_6^1} + \|\tilde{\Psi}\|_{W_3^1} + \|\nabla\tilde{\Psi}\|_{L_{3/2}} \leqq \epsilon. \tag{5.11}$$

Given $U(t) \in \mathbb{J}_\epsilon$, we start with the estimate of $\Xi_U(s)$, which is defined by (5.5). Below, we use the inequalities: $\epsilon^k \leqq \epsilon$ and $|||U|||_t^k \leqq |||U|||_t$ for $k \geqq 2$, which follows from the fact that $|||U|||_t \leqq \epsilon \leqq 1$. By (4.4) with $p = 3/2$ and (1.22), we have

$$\big\|\Xi_U(s)\big\|_{L_3} + \big\|\nabla \Xi_U(s)\big\|_{L_{3/2}} \leqq C\big\|\mathcal{F}_3\big(U(s)\big)\big\|_{L_{3/2}}. \tag{5.12}$$

Below, we set $f_\ell(t) = f_{\ell,\gamma}(t)$, because γ has been fixed, where $f_{\ell,\gamma}(t)$ is a function defined in (1.51). To get the estimate of the right-hand side of (5.12), we use the following estimates obtained by Hölder's inequality and the definition of $f_\ell(s)$:

$$\begin{aligned}
\big\|\tilde a \nabla u(s)\big\|_{L_{3/2}} &\leqq \|\tilde a\|_{L_3}\big\|\nabla u(s)\big\|_{L_3} \leqq \|\tilde a\|_{L_3}[\nabla u]_{3,1/2,\gamma,s} f_{-1/2}(s) \qquad (5.13)\\
\big\|u(s)\nabla \tilde a\big\|_{L_{3/2}} &\leqq \big\|u(s)\big\|_{L_3}\|\nabla \tilde a\|_{L_3} \leqq \|\nabla \tilde a\|_{L_3}[u]_{3,0,\gamma,s} f_0(s),\\
\big\|u(s)\nabla v(s)\big\|_{L_{3/2}} &\leqq \big\|u(s)\big\|_{L_3}\big\|\nabla v(s)\big\|_{L_3} \leqq [u]_{3,0,\gamma,s}[\nabla v]_{3,1/2,\gamma,s} f_{-1/2}(s),\\
\big\|\tilde a \tilde b u(s)\big\|_{L_{3/2}} &\leqq \|\tilde a\|_{L_3}\|\tilde b\|_{L_6}\big\|u(s)\big\|_{L_6} \leqq \|\tilde a\|_{L_3}\|\tilde b\|_{L_6}[u]_{6,1/4,\gamma,s} f_{-1/4}(s),\\
\big\|\tilde a u(s) v(s)\big\|_{L_{3/2}} &\leqq \|\tilde a\|_{L_6}\big\|u(s)\big\|_{L_3}\big\|v(s)\big\|_{L_6} \leqq \|\tilde a\|_{L_6}[u]_{3,0,\gamma,s}[v]_{6,1/4,\gamma,s} f_{-1/4}(s),\\
\big\|u(s)v(s)w(s)\big\|_{L_{3/2}} &\leqq \big\|u(s)\big\|_{L_3}\big\|v(s)\big\|_{L_6}\big\|w(s)\big\|_{L_6}\\
&\leqq [u]_{3,0,\gamma,s}[v]_{6,1/4,\gamma,s}[w]_{6,1/4,\gamma,s} f_{-1/2}(s). \qquad (5.14)
\end{aligned}$$

Noting that $f_0(s) \leqq f_{-1/4}(s) \leqq f_{-1/2}(s)$, by (5.14), (5.6), (5.11) and (5.12) we have

$$\big\|\Xi_U(s)\big\|_{L_3} + \big\|\nabla \Xi_U\big\|_{L_{3/2}} + \big\|\mathcal{F}_3\big(U(s)\big)\big\|_{L_{3/2}} \leqq C f_{-1/2}(s)\big(\epsilon + |||U|||_s\big)|||U|||_s. \tag{5.15}$$

To estimate $\Phi_j U(t)$, $j = 1, 2$, we use the following estimates:

$$\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\ell} f_{-m}(s)\,ds \leqq C f_{1-\ell-m}(t) \tag{5.16}$$

for $0 \leqq \ell, m < 1$. By (5.1), (5.2), (5.15) and (5.16), we have

$$\begin{aligned}
\big\|\Phi_1 U(t)\big\|_{L_3} &\leqq C_{3,3}\int_0^t e^{-2\gamma(t-s)} f_{-1/2}(s)\,ds\big(\epsilon + |||U|||_t\big)|||U|||_t\\
&\leqq C f_{1/2}(t)\big(\epsilon + |||U|||_t\big)|||U|||_t.
\end{aligned}$$

In the similar manner, we have also

$$\begin{aligned}
\big\|\Phi_1 U(t)\big\|_{L_6} &\leqq C_{3,6}\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\frac{3}{2}(\frac{1}{3}-\frac{1}{6})} f_{-1/2}(s)\,ds\big(\epsilon + |||U|||_t\big)|||U|||_t\\
&\leqq C f_{1/4}(t)\big(\epsilon + |||U|||_t\big)|||U|||_t;
\end{aligned}$$

$$\begin{aligned}\big\|\nabla\Phi_1 U(t)\big\|_{L_3} &\leqq C_{3,3}\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\frac{1}{2}} f_{-1/2}(s)\,ds\big(\epsilon+|||U|||_t\big)|||U|||_t\\ &\leqq Cf_0(t)\big(\epsilon+|||U|||_t\big)|||U|||_t.\end{aligned}$$

Noting that $f_\ell(t)f_m(t)\leqq f_{\ell+m}(t)$ and combining above three estimates, we have

$$|||\Phi_1 U|||_t\leqq C\min\big(t^{1/2},1\big)\big(\epsilon+|||U|||_t\big)|||U|||_t. \tag{5.17}$$

In particular, by (5.17) we have

$$\lim_{t\to 0+}|||\Phi_1 U|||_t=0,\qquad |||\Phi_1 U|||_t\leqq C\big(\epsilon+|||U|||_t\big)|||U|||_t. \tag{5.18}$$

Now, we shall estimate $|||\Phi_2 U|||_t$. To do this, first we shall estimate the L_2 norm of $G_2(U(s))$ and $G_3(U(s))$. By (4.2), (4.3), (5.6), (5.11) and (5.15) we have

$$\begin{aligned}\big\|\tilde a\,\Xi_U(s)\big\|_{L_2} &\leqq \|\tilde a\|_{L_6}\big\|\Xi_U(s)\big\|_{L_3}\leqq C\|\tilde a\|_{L_6}\big(\epsilon+|||U|||_s\big)|||U|||_s f_{-1/2}(s),\\ \big\|u(s)\tilde\Psi\big\|_{L_2} &\leqq \big\|u(s)\big\|_{L_6}\|\tilde\Psi\|_{L_3}\leqq\|\tilde\Psi\|_{L_3}[u]_{6,1/4,\gamma,s}f_{-1/4}(s),\\ \big\|u(s)\Xi_U(s)\big\|_{L_2} &\leqq \big\|u(s)\big\|_{L_6}\big\|\Xi_U(s)\big\|_{L_3}\leqq C\big(\epsilon+|||U|||_s\big)|||U|||_s[u]_{6,1/4,\gamma,s}f_{-3/4}(s),\end{aligned}$$

and therefore by (5.6) and (5.11) we have

$$\big\|G_2\big(U(s)\big)\big\|_{L_2}\leqq C\big(\epsilon+|||U|||_s\big)\big[|||U|||_s f_{-1/2}(s)+[U]_{6,1/4,\gamma,s}f_{-3/4}(s)\big]. \tag{5.19}$$

And also, we have

$$\begin{aligned}\big\|\tilde a\nabla u(s)\big\|_{L_2} &\leqq \|\tilde a\|_{L_6}\big\|\nabla u(s)\big\|_{L_3}\leqq\|\tilde a\|_{L_6}[\nabla u]_{3,1/2,\gamma,s}f_{-1/2}(s),\\ \big\|u(s)\nabla\tilde a\big\|_{L_2} &\leqq \big\|u(s)\big\|_{L_6}\|\nabla\tilde a\|_{L_3}\leqq\|\nabla\tilde a\|_{L_3}[\nabla u]_{6,1/4,\gamma,s}f_{-1/4}(s),\\ \big\|u(s)\nabla v(s)\big\|_{L_2} &\leqq \big\|u(s)\big\|_{L_6}\big\|\nabla v(s)\big\|_{L_3}\leqq[u]_{6,1/4,\gamma,s}[\nabla u]_{3,1/2,\gamma,s}f_{-3/4}(s),\\ \big\|\tilde a u(s)\big\|_{L_2} &\leqq \|\tilde a\|_{L_3}\big\|u(s)\big\|_{L_6}\leqq\|\tilde a\|_{L_3}[u]_{6,1/4,\gamma,s}f_{-1/4}(s),\\ \big\|u(s)v(s)\big\|_{L_2} &\leqq \big\|u(s)\big\|_{L_6}\big\|v(s)\big\|_{L_3}\leqq[\nabla u]_{6,1/4,\gamma,s}[v]_{3,0,\gamma,s}f_{-1/4}(s),\\ \big\|\tilde a\tilde b u(s)\big\|_{L_2} &\leqq \|\tilde a\|_{L_6}\|\tilde b\|_{L_6}\big\|u(s)\big\|_{L_6}\leqq\|\tilde a\|_{L_6}\|\tilde b\|_{L_6}[u]_{6,1/4,\gamma,s}f_{-1/4}(s),\\ \big\|\tilde a u(s)v(s)\big\|_{L_2} &\leqq \|\tilde a\|_{L_6}\big\|u(s)\big\|_{L_6}\big\|v(s)\big\|_{L_6}\leqq\|\tilde a\|_{L_6}[u]_{6,1/4,\gamma,s}[v]_{6,1/4,\gamma,s}f_{-1/2}(s),\\ \big\|u(s)v(s)w(s)\big\|_{L_2} &\leqq \big\|u(s)\big\|_{L_6}\big\|v(s)\big\|_{L_6}\big\|w(s)\big\|_{L_6}\leqq\epsilon[u]_{6,1/4,\gamma,s}[v]_{6,1/4,\gamma,s}[w]_{6,1/4,\gamma,s}f_{-3/4}(s)\end{aligned}$$

and therefore by (5.6) and (5.11) we have

$$\big\|G_3\big(U(s)\big)\big\|_{L_2}\leqq C\big(\epsilon+|||U|||_s\big)\big[|||U|||_s f_{-1/2}(s)+[U]_{6,1/4,\gamma,s}f_{-3/4}(s)\big]. \tag{5.20}$$

To obtain the estimate

$$|||\Phi_2 U|||_t \leqq C\big(\epsilon + |||U|||_t\big)\big([U]_{6,1/4,\gamma,t} + |||U|||_t \min\big(t^{1/4}, 1\big)\big), \tag{5.21}$$

using (5.19), (5.20), (5.16), (5.1) and (5.2), we observe the following estimates:

$$\begin{aligned}
\big\|\Phi_2 U(t)\big\|_{L_3} &\leqq C_{2,3}\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{3})}\big[f_{-3/4}(s)[U]_{6,1/4,\gamma,t} + f_{-1/2}(s)|||U|||_t\big]\,ds \\
&\quad \times \big(\epsilon + |||U|||_t\big) \\
&\leqq C\big(\epsilon + |||U|||_t\big)\big(f_0(t)[U]_{6,1/4,\gamma,t} + f_{1/4}(t)|||U|||_t\big); \\
\big\|\Phi_2 U(t)\big\|_{L_6} &\leqq C_{2,6}\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{6})}\big[f_{-3/4}(s)[U]_{6,1/4,\gamma,t} + f_{-1/2}(s)|||U|||_t\big]\,ds \\
&\quad \times \big(\epsilon + |||U|||_t\big) \\
&\leqq C\big(\epsilon + |||U|||_t\big)\big(f_{-1/4}(t)[U]_{6,1/4,\gamma,t} + f_0(t)|||U|||_t\big); \\
\big\|\nabla\Phi_2 U(t)\big\|_{L_3} &\leqq C_{2,3}\int_0^t e^{-2\gamma(t-s)}(t-s)^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{3})-\frac{1}{2}}\big[f_{-3/4}(s)[U]_{6,1/4,\gamma,t} + f_{-1/2}(s)|||U|||_t\big]\,ds \\
&\quad \times \big(\epsilon + |||U|||_t\big) \\
&\leqq C\big(\epsilon + |||U|||_t\big)\big(f_{-1/2}(t)[U]_{6,1/4,\gamma,t} + f_{-1/4}(t)|||U|||_t\big).
\end{aligned}$$

By (5.21) and (5.7), we have

$$\lim_{t\to 0+} |||\Phi_2 U|||_t = 0, \qquad |||\Phi_2 U|||_t \leqq C\big(\epsilon + |||U|||_t\big)|||U|||_t. \tag{5.22}$$

Therefore, by (5.6), Lemma 5.1, (5.18) and (5.22) we have

$$\lim_{t\to 0+}\big\|\Phi U(t) - U_0\big\|_{L_3} = 0, \qquad \lim_{t\to 0+}\big([\Phi U]_{6,1/4,\gamma,t} + [\nabla\Phi U]_{3,1/2,\gamma,t}\big) = 0, \tag{5.23}$$

$$|||\Phi U|||_t \leqq C\|U_0\|_{L_3} + 2C\epsilon^2. \tag{5.24}$$

If we choose $\epsilon \in (0,1)$ in such a way that

$$C\|U_0\|_{L_3} \leqq \epsilon/2, \qquad 2C\epsilon \leqq 1/2, \tag{5.25}$$

by (5.24) we have

$$|||\Phi U|||_t \leqq \epsilon, \quad \forall t > 0, \tag{5.26}$$

which implies that Φ is a map from $\mathbb{J}_\epsilon$ into itself.

Given $U_1, U_2 \in \mathbb{J}_\epsilon$, applying the same argument to $\Phi_j U_1 - \Phi_j U_2$, $j = 1, 2$, we have

$$|||\Phi U_1 - \Phi U_2|||_t \leqq C\big(\epsilon + |||U_1|||_t + |||U_2|||_t\big)|||U_1 - U_2|||_t \leqq 3C\epsilon|||U_1 - U_2|||_t, \tag{5.27}$$

where we have used the fact that $|||U_j|||_t \leqq \epsilon \leqq 1$ to have $|||U_j|||_t^2 \leqq |||U_j|||_t$ $(j = 1, 2)$. Therefore, if we choose $\epsilon \in (0, 1)$ in such a way that $3C\epsilon \leqq 1/2$, by (5.27) we have

$$|||\Phi U_1 - \Phi U_2|||_t \leqq \frac{1}{2}|||U_1 - U_2|||_t \tag{5.28}$$

for any $U_1, U_2 \in \mathbb{J}_\epsilon$, which implies that Φ is a contraction map on $\mathbb{J}_\epsilon$. Since $\mathbb{J}_\epsilon$ is a complete metric space endowed with the metric induced by the norm $\sup\{|||\ |||_t;\ t > 0\}$, there exists a unique fixed point $U \in \mathbb{J}_\epsilon$ such that $U = \Phi U$, which solves (5.4).

Now, we shall estimate $\|U(t)\|_{L_\infty}$. By (5.1), (5.2), (5.15), (5.16), (5.19) and (5.20) we have

$$\begin{aligned}
\big\|T(t)U_0\big\|_{L_\infty} &\leqq C_{3,\infty} f_{-1/2}(t)\|U_0\|_{L_3};\\
\big\|\Phi_1 U(t)\big\|_{L_\infty} &\leqq C_{3,\infty}\int_0^t e^{-2\gamma(t-s)}(t-s)^{-1/2} f_{-1/2}(s)\,ds\big(\epsilon + |||U|||_t\big)|||U|||_t\\
&\leqq C f_0(t)\big(\epsilon + |||U|||_t\big)|||U|||_t;\\
\big\|\Phi_2 U(t)\big\|_{L_\infty} &\leqq C_{2,\infty}\int_0^t e^{-2\gamma(t-s)}(t-s)^{-3/4}\big[f_{-1/2}(s)|||U|||_t + f_{-3/4}(s)[U]_{6,1/4,\gamma,t}\big]\,ds\\
&\quad\times\big(\epsilon + |||U|||_t\big)\\
&\leqq C\big(f_{-1/4}(t)|||U|||_t + f_{-1/2}(t)[U]_{6,1/4,\gamma,t}\big)\big(\epsilon + |||U|||_t\big).
\end{aligned} \tag{5.29}$$

Since $U(t) = \Phi U(t)$, it follows from (5.29) that

$$[U]_{\infty,1/2,\gamma,t} \leqq C\epsilon. \tag{5.30}$$

Finally, we discuss the estimate of Ξ_U. By (5.15) we have

$$[\Xi_U]_{3,1/2,\gamma,t} + [\nabla\Xi_U]_{3/2,1/2,\gamma,t} \leqq C\epsilon^2. \tag{5.31}$$

To obtain

$$[\nabla\Xi_U]_{3,1,\gamma,t} \leqq C\epsilon^2, \tag{5.32}$$

in view of (1.22) we shall estimate $\|\mathcal{F}_3(U(t))\|_{L_3}$. By (4.2) and Hölder's inequality, we have

$$\begin{aligned}
\big\|\tilde a\nabla u(t)\big\|_{L_3} &\leqq \|\tilde a\|_{L_\infty}\big\|\nabla u(t)\big\|_{L_3} \leqq C\|\tilde a\|_{W_6^1}[\nabla u]_{3,1/2,\gamma,t} f_{-1/2}(t),\\
\big\|u(t)\nabla v(t)\big\|_{L_3} &\leqq \big\|u(t)\big\|_{L_\infty}\big\|\nabla v(t)\big\|_{L_3} \leqq C[u]_{\infty,1/2,\gamma,t}[\nabla v]_{3,1/2,\gamma,t} f_{-1}(t),\\
\big\|\tilde a u(t)\big\|_{L_3} &\leqq \|\tilde a\|_{L_\infty}\big\|u(t)\big\|_{L_3} \leqq C\|\tilde a\|_{W_6^1}[u]_{3,0,\gamma,t} f_{-1/2}(t),
\end{aligned}$$

$$\begin{aligned}
\|u(t)v(t)\|_{L_3} &\leqq \|u(t)\|_{L_\infty}\|v(t)\|_{L_3} \leqq C[u]_{\infty,1/2,\gamma,t}[v]_{3,0,\gamma,t}f_{-1/2}(t),\\
\|\tilde{a}\tilde{b}u(t)\|_{L_3} &\leqq \|\tilde{a}\|_{L_\infty}\|\tilde{b}\|_{L_\infty}\|u(t)\|_{L_3} \leqq C\|\tilde{a}\|_{W_6^1}\|\tilde{b}\|_{W_6^1}[u]_{3,0,\gamma,t}f_0(t),\\
\|\tilde{a}u(t)v(t)\|_{L_3} &\leqq \|\tilde{a}\|_{L_\infty}\|u(t)\|_{L_\infty}\|v(t)\|_{L_3} \leqq C\|\tilde{a}\|_{W_6^1}[u]_{\infty,1/2,\gamma,t}[v]_{3,0,\gamma,t}f_{-1/2}(t),\\
\|u(t)v(t)w(t)\|_{L_3} &\leqq \|u(t)\|_{L_\infty}\|v(t)\|_{L_\infty}\|w(t)\|_{L_3} \leqq C[u]_{\infty,1/2,\gamma,t}[v]_{\infty,1/2,\gamma,t}[w]_{3,0,\gamma,t}f_{-1}(t),
\end{aligned}$$

and therefore by (5.11), (5.26) and (5.30) we have

$$\|\mathcal{F}_3(U(t))\|_{L_3} \leqq Cf_{-1}(t)\epsilon^2,$$

which combined with (1.22) implies (5.32). This completes the proof of Theorem 1.3.

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Appendix A. A proof of (4.4)

In this appendix, we shall prove that

$$\|u\|_{L_{3p/(3-p)}(\Omega)} \leqq C\|\nabla u\|_{L_p(\Omega)} \tag{A.1}$$

for any $u \in \hat{W}_p^1(\Omega)$. Since the boundary is assumed to be a $C^{2,1}$ hypersurface, given $u \in \hat{W}_p^1(\Omega)$ there exists an extension $v \in \hat{W}_p^1(\mathbb{R}^3)$ of u such that

$$\|\nabla v\|_{L_p(\mathbb{R}^3)} \leqq C\{\|\nabla u\|_{L_p(\Omega)} + \|u\|_{L_p(\Omega\cap B_R)}\} \tag{A.2}$$

for some large R such that $B_R \supset \Omega$ when Ω is a bounded domain, $B_R \supset \mathbb{R}^n \setminus \Omega$ when Ω is an exterior domain and $\Omega \setminus B_R = \mathbb{R}^3_+ \setminus B_R$ when Ω is a perturbed half space. Since

$$\int_{\Omega_0} u\,dx = 0,$$

by the Poincaré inequality we know that

$$\|u\|_{L_p(\Omega\cap B_R)} \leqq C\|\nabla u\|_{L_p(\Omega)},$$

which combined with (A.2) implies that

$$\|\nabla v\|_{L_p(\mathbb{R}^3)} \leqq C\|\nabla u\|_{L_p(\Omega)}. \tag{A.3}$$

As we know well (cf. Farwig and Sohr [3]), there exists a sequence $\{v_j\} \subset C_0^\infty(\mathbb{R}^3)$ such that

$$\lim_{j\to\infty}\|\nabla(v_j - v)\|_{L_p(\mathbb{R}^3)} = 0. \tag{A.4}$$

By the Sobolev inequality (cf. Galdi [6, Lemma 2,2, II]), we know that

$$\|v_j\|_{L_{3p/(3-p)}(\mathbb{R}^3)} \leqq C\|\nabla v_j\|_{L_p(\mathbb{R}^3)}, \tag{A.5}$$

$$\|v_j - v_k\|_{L_{3p/(3-p)}(\mathbb{R}^3)} \leqq C\left\|\nabla(v_j - v_k)\right\|_{L_p(\mathbb{R}^3)}. \tag{A.6}$$

By (A.4)–(A.6), we see that $v \in L_{3p/(3-p)}(\mathbb{R}^n)$ and

$$\|v\|_{L_{3p/(3-p)}(\mathbb{R}^3)} \leqq C\|\nabla v\|_{L_p(\mathbb{R}^3)}. \tag{A.7}$$

Since $u = v$ in Ω, by (A.7) we have

$$\|u\|_{L_{3p/(3-p)}(\Omega)} = \|v\|_{L_{3p/(3-p)}(\Omega)} \leqq \|v\|_{L_{3p/(3-p)}(\mathbb{R}^3)} \leqq C\|\nabla v\|_{L_p(\mathbb{R}^3)},$$

which combined with (A.3) implies (A.1).

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